

# The algebraic de Rham theorem for toric varieties

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Dedicated to Professor Takeshi Kotake on his sixtieth birthday

## Abstract

On an arbitrary toric variety, we introduce the logarithmic double complex, which is essentially the same as the algebraic de Rham complex in the nonsingular case, but which behaves much better in the singular case.

Over the field of complex numbers, we prove the toric analog of the algebraic de Rham theorem which Grothendieck formulated and proved for general nonsingular algebraic varieties re-interpreting an earlier work of Hodge-Atiyah. Namely, for a finite simplicial fan which need not be complete, the complex cohomology groups of the corresponding toric variety as an analytic space coincide with the hypercohomology groups of the single complex associated to the logarithmic double complex. They can then be described combinatorially as Ishida's cohomology groups for the fan.

We also prove vanishing theorems for Ishida's cohomology groups. As a consequence, we deduce directly that the complex cohomology groups vanish in odd degrees for toric varieties which correspond to finite simplicial fans with full-dimensional convex support. In the particular case of complete simplicial fans, we thus have a direct proof for an earlier result of Danilov and the author.

## Introduction

Let  $\Delta$  be a finite fan for a free  $\mathbf{Z}$ -module  $N$  of rank  $r$ , and denote by  $X := T_N \text{emb}(\Delta)$  the associated  $r$ -dimensional toric variety over the field  $\mathbf{C}$  of complex numbers. We also denote by  $\mathcal{X} := X^{\text{an}}$  the associated complex analytic space.

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In Section 1, we introduce the *logarithmic double complex*  $\mathcal{L}_X^\bullet$  of  $\mathcal{O}_X$ -modules. The associated single complex  $\mathcal{L}_X^\bullet$ , called the *logarithmic complex* for  $X$ , turns out to be a natural toric generalization of the algebraic de Rham complex for general nonsingular algebraic varieties. Indeed, when  $X$  is a nonsingular toric variety, then  $\mathcal{L}_X^\bullet$  is canonically quasi-isomorphic to the algebraic de Rham complex  $\Omega_X^\bullet$ . More generally, suppose  $\Delta$  is a simplicial fan, so that  $X$  has at worst quotient singularities. Then  $\mathcal{L}_X^\bullet$  is canonically quasi-isomorphic to the complex  $\tilde{\Omega}_X^\bullet$  of  $\mathcal{O}_X$ -modules consisting of the Zariski differential forms on  $X$ . By Danilov's Poincaré lemma [2] (see also Ishida [13, Prop. 2.1]), we thus see that the associated complex analytic complex  $(\mathcal{L}_X^\bullet)^{\text{an}}$  is quasi-isomorphic to the constant sheaf  $\mathbf{C}_X$ . We would like to point out here that according to Ishida,  $(\mathcal{L}_X^\bullet)^{\text{an}}[2r]$  is quasi-isomorphic to the *globally normalized dualizing complex* of  $\mathbf{C}_X$ -modules in the sense of Verdier [24].

In Section 2, we recall the definition of Ishida's  $p$ -th complex  $C^\bullet(\Delta, \Lambda^p)$  of  $\mathbf{Z}$ -modules for  $0 \leq p \leq r$  and its cohomology groups  $H^q(\Delta, \Lambda^p)$ , which were considered in Ishida [12] and [16] in different notation.

Section 3 is devoted to the algebraic de Rham theorem for the logarithmic complex  $\mathcal{L}_X^\bullet$ . Without any condition on the fan  $\Delta$ , we first show that the algebraic hypercohomology group  $\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)$  is a direct sum of the scalar extensions  $H^\bullet(\Delta, \Lambda^p)_{\mathbf{C}}$  to  $\mathbf{C}$  of Ishida's cohomology groups for various  $p$ 's.

When  $\Delta$  is simplicial but not necessarily complete, we mimic the proof, due to Grothendieck [11], of the usual algebraic de Rham theorem to show that the algebraic hypercohomology group  $\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)$  is canonically isomorphic to the complex analytic hypercohomology group  $\mathbf{H}^\bullet(\mathcal{X}, (\mathcal{L}_X^\bullet)^{\text{an}})$ . Consequently, we have our main result

$$H^l(\mathcal{X}, \mathbf{C}) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for each } l$$

for  $\Delta$  simplicial but not necessarily complete.

In Section 4, we prove various vanishing theorems for Ishida's cohomology groups. When  $\Delta$  is simplicial and complete, we have a direct combinatorial proof for

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \quad \text{for } q \neq p.$$

Hence  $H^l(\mathcal{X}, \mathbf{C}) = 0$  for  $l$  odd, while

$$H^{2p}(\mathcal{X}, \mathbf{C}) = H^p(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for } 0 \leq p \leq r.$$

In the complete nonsingular case, this was proved earlier by means of the usual algebraic de Rham theorem by Danilov [2] and [16, Thm. 3.11].

When  $\Delta$  is simplicial with support  $|\Delta|$  convex of dimension  $r$  but not necessarily complete, we continue to have the same vanishing theorem

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \quad \text{for } q \neq p,$$

which turns out to be a special case of a more general vanishing theorem due to Ishida. In view of our algebraic de Rham theorem in Section 3, we continue to have  $H^l(\mathcal{X}, \mathbf{C}) = 0$  for  $l$  odd, while

$$H^{2p}(\mathcal{X}, \mathbf{C}) = H^p(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for } 0 \leq p \leq r$$

when  $\Delta$  is simplicial with  $|\Delta|$  convex of dimension  $r$  but not necessarily complete.

The intersection cohomology theory of Goresky and MacPherson [8], [10], which was further developed by Beilinson, Bernstein and Deligne [1], attaches to singular spaces new topological invariants much better behaved than ordinary cohomology or homology groups. Since a toric variety is a simple-minded singular space described in terms of a fan, it is natural to try to describe its intersection cohomology group and, more generally its intersection complex, in terms of the fan as well. We believe that the logarithmic complex will play a key role in describing the intersection cohomology group (especially with respect to the middle perversity) of toric varieties.

There have been earlier attempts in this direction by J. N. Bernstein, A. G. Khovanskii and R. D. MacPherson (see Stanley [22]), Denef and Loeser [6], Fieseler [7] and others. However, they depend either on the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber (see Beilinson, Bernstein and Deligne [1] as well as Goresky and MacPherson [9]) or on the purity theorem of Deligne and Gabber (cf. Deligne [4], [5]).

This problem turns out to be closely related to the problem of finding an elementary proof of the strong Lefschetz theorem for toric varieties, and hence of the elementary proof due to Stanley [21] of the “ $g$ -theorem” for simplicial convex polytopes conjectured earlier by McMullen [14]. (See also [17], [18], Stanley [23] and McMullen [15].)

We refer the reader to [16] for basic results on toric varieties used in this paper.

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## 1 Logarithmic double complex

Let  $\Delta$  be a finite fan for a free  $\mathbf{Z}$ -module  $N$  of rank  $r$ , and denote by  $X := T_N \text{emb}(\Delta)$  the associated  $r$ -dimensional toric variety over the field  $\mathbf{C}$  of complex numbers. The complement  $D := X \setminus T_N$  of the algebraic torus  $T_N := N \otimes_{\mathbf{Z}} \mathbf{C}^* \cong (\mathbf{C}^*)^r$  is a Weil divisor on  $X$  but is not necessarily a Cartier divisor. The dual  $\mathbf{Z}$ -module  $M := \text{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ , with the canonical bilinear pairing  $\langle \ , \ \rangle : M \times N \rightarrow \mathbf{Z}$ , is isomorphic to the character group of the algebraic torus  $T_N$ . For each  $m \in M$  we denote the corresponding character by  $t^m : T_N \rightarrow \mathbf{C}^*$  (which was denoted by  $\mathbf{e}(m)$  in [16]), and identify  $\mathbf{C}[M] := \bigoplus_{m \in M} \mathbf{C} t^m$  with the group algebra of  $M$  over  $\mathbf{C}$ . Hence  $T_N$  is the group of  $\mathbf{C}$ -valued points of the group scheme  $\text{Spec}(\mathbf{C}[M])$ .

Each  $n \in N$  gives rise to a  $\mathbf{C}$ -derivation  $\delta_n$  of  $\mathbf{C}[M]$  defined by  $\delta_n(t^m) := \langle m, n \rangle t^m$ . Consequently, we have a canonical isomorphism to the Lie algebra

$$\mathbf{C} \otimes_{\mathbf{Z}} N \xrightarrow{\sim} \text{Lie}(T_N), \quad 1 \otimes n \mapsto \delta_n,$$

hence an  $\mathcal{O}_X$ -isomorphism  $\mathcal{O}_X \otimes_{\mathbf{Z}} N \xrightarrow{\sim} \Theta_X(-\log D)$ , where the right hand side is the sheaf of germs of algebraic vector fields on  $X$  with logarithmic zeros along the Weil divisor  $D$ . Its dual  $\Omega_X^1(\log D)$  is the sheaf of germs of algebraic 1-forms with logarithmic poles along  $D$ , and we get an  $\mathcal{O}_X$ -isomorphism

$$\mathcal{O}_X \otimes_{\mathbf{Z}} M \xrightarrow{\sim} \Omega_X^1(\log D), \quad 1 \otimes m \mapsto \frac{dt^m}{t^m}.$$

Taking the exterior product, we thus get an  $\mathcal{O}_X$ -isomorphism  $\mathcal{O}_X \otimes_{\mathbf{Z}} \bigwedge^p M \xrightarrow{\sim} \Omega_X^p(\log D)$ . The exterior differentiation  $d$  on the right hand side corresponds to the operation on the left hand side which sends a  $p$ -form  $t^m \otimes m_1 \wedge \cdots \wedge m_p$  to the  $(p+1)$ -form  $t^m \otimes m \wedge m_1 \wedge \cdots \wedge m_p$  (cf. [16, Chap. 3]).

Recall that the set of  $T_N$ -orbits in  $X$  is in one-to-one correspondence with  $\Delta$  by the map which sends each  $\sigma \in \Delta$  to the  $T_N$ -orbit

$$\text{orb}(\sigma) = \text{Spec}(\mathbf{C}[M \cap \sigma^\perp]) = T_{N/\mathbf{Z}(N \cap \sigma)}.$$

The closure  $V(\sigma)$  in  $X$  of  $\text{orb}(\sigma)$  is known to be a toric variety with respect to a fan for the  $\mathbf{Z}$ -module  $N/\mathbf{Z}(N \cap \sigma)$ . Namely,

$$V(\sigma) = T_{N/\mathbf{Z}(N \cap \sigma)} \text{emb}(\{(\tau + (-\sigma))/\mathbf{R}\sigma \mid \tau \in \text{Star}_\sigma(\Delta)\}),$$

where  $\mathbf{R}\sigma = \sigma + (-\sigma)$  is the smallest  $\mathbf{R}$ -subspace containing  $\sigma$  of  $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ , while  $\text{Star}_\sigma(\Delta) := \{\tau \in \Delta \mid \tau \succ \sigma\}$ . Hence  $D(\sigma) := V(\sigma) \setminus \text{orb}(\sigma)$  is a Weil divisor on  $V(\sigma)$ . In particular, we have  $\text{orb}(\{0\}) = T_N$ ,  $D(\{0\}) = D$  and  $V(\{0\}) = X$ .

For each integer  $q$  with  $0 \leq q \leq r$ , denote  $\Delta(q) := \{\sigma \in \Delta \mid \dim \sigma = q\}$ . For each pair of integers  $p, q$ , let

$$\mathcal{L}_X^{p,q} := \bigoplus_{\sigma \in \Delta(q)} \Omega_{V(\sigma)}^{p-q}(\log D(\sigma)) = \bigoplus_{\sigma \in \Delta(q)} \mathcal{O}_{V(\sigma)} \otimes_{\mathbf{Z}} \bigwedge^{p-q}(M \cap \sigma^\perp) \quad \text{if } 0 \leq q \leq p,$$

and  $\mathcal{L}^{p,q} = 0$  otherwise.  $d_I : \mathcal{L}_X^{p,q} \rightarrow \mathcal{L}_X^{p+1,q}$  is defined to be the direct sum of the exterior differentiation for each  $\sigma \in \Delta(q)$ . We define  $d_{II} : \mathcal{L}_X^{p,q} \rightarrow \mathcal{L}_X^{p,q+1}$  as follows: The  $(\sigma, \tau)$ -component of  $d_{II}$  for  $\sigma \in \Delta(q)$  and  $\tau \in \Delta(q+1)$  is defined to be zero when  $\sigma$  is not a face of  $\tau$ . On the other hand, if  $\sigma$  is a face of  $\tau$ , then a primitive element  $n \in N$  is uniquely determined modulo  $N \cap \mathbf{R}\sigma$  so that  $\tau + (-\sigma) = \mathbf{R}_{\geq 0}n + \mathbf{R}\sigma$ .  $M \cap \tau^\perp$  is a  $\mathbf{Z}$ -submodule of corank one in  $M \cap \sigma^\perp$ . The  $(\sigma, \tau)$ -component of  $d_{II}$  in this case is then defined to

be the tensor product of the restriction homomorphism  $\mathcal{O}_{V(\sigma)} \rightarrow \mathcal{O}_{V(\tau)}$  with the interior product with respect to  $n$ . Namely, an element in the  $\sigma$ -component of  $\mathcal{L}_X^{p,q}$  of the form

$$t^m \otimes m_1 \wedge m_2 \wedge \cdots \wedge m_{p-q}, \quad m, m_1 \in M \cap \sigma^\perp, \quad m_2, \dots, m_{p-q} \in M \cap \tau^\perp$$

is sent to  $t^m \otimes \langle m_1, n \rangle m_2 \wedge \cdots \wedge m_{p-q}$  if  $m \in M \cap \tau^\perp$ , and to 0 otherwise.  $d_{\text{II}}$  is the Poincaré residue map.

$d_{\text{I}} \circ d_{\text{I}} = 0$  and  $d_{\text{I}} \circ d_{\text{II}} + d_{\text{II}} \circ d_{\text{I}} = 0$  are obvious, while  $d_{\text{II}} \circ d_{\text{II}} = 0$  was shown in Ishida [12, Lemma 1.4 and Prop. 1.6]. Consequently, we get a double complex  $\mathcal{L}_X^{\bullet, \bullet}$  of  $\mathcal{O}_X$ -modules, which we call the *logarithmic double complex* for the toric variety  $X$ . The associated single complex is denoted by  $\mathcal{L}_X^\bullet$  and is called the *logarithmic complex* for  $X$ .

For simplicity, we denote by  $\mathcal{X} := X^{\text{an}}$  the complex analytic space associated to the toric variety  $X = T_N \text{emb}(\Delta)$ .

**THEOREM 1.1** *If  $\Delta$  is a simplicial fan, then we have a quasi-isomorphism  $\mathbf{C}_{\mathcal{X}} \simeq (\mathcal{L}_X^\bullet)^{\text{an}}$ .*

**PROOF.** It suffices to prove the assertion when  $X$  is affine, and there is an easy proof in that case. However, we here give a direct proof valid for general  $X$ .

Let  $\tilde{\Omega}_X^\bullet := j_* \Omega_U^\bullet$  be the complex of  $\mathcal{O}_X$ -modules consisting of the Zariski differential forms on  $X$ , where  $j : U \rightarrow X$  is the open immersion of the smooth locus  $U$  of  $X$ . By Danilov's Poincaré lemma [2] (see also Ishida [13, Prop. 2.1]), we have a quasi-isomorphism  $\mathbf{C}_{\mathcal{X}} \simeq (\tilde{\Omega}_X^\bullet)^{\text{an}}$  for an *arbitrary* fan  $\Delta$ . Furthermore, if  $\Delta$  is simplicial, then by [16, Theorem 3.6] we have a quasi-isomorphism  $\tilde{\Omega}_X^p \simeq \mathcal{L}_X^{p, \bullet}$  for each fixed  $p$ . q.e.d.

**REMARK.** In [16],  $\mathcal{L}_X^{p, \bullet}$  was denoted by  $\mathcal{K}^\bullet(X; p)$ .

As for an *arbitrary* fan  $\Delta$  for  $N \cong (\mathbf{Z})^r$  which need not be simplicial, Ishida has a proof for the following amazing result:  $(\mathcal{L}_X^\bullet)^{\text{an}}[2r]$  is quasi-isomorphic to the *globally normalized dualizing complex*  $\mathcal{D}_X^\bullet$  of  $\mathbf{C}_{\mathcal{X}}$ -modules in the sense of Verdier [24]. The point of this result of Ishida's lies in the fact that the dualizing complex  $\mathcal{D}_X^\bullet$  can be expressed in terms of a complex comprizing of *algebraic* and *coherent*  $\mathcal{O}_X$ -modules. Analogously, Ishida [12, Theorem 3.3] and [13, Theorem 5.4] earlier showed  $\mathcal{L}_X^{r, \bullet}[r]$  to be quasi-isomorphic to the *globally normalized dualizing complex* of  $\mathcal{O}_X$ -modules.

The following is a toric generalization of a result due to Grothendieck and Deligne [3, I, §3 and §6] on the complement in a smooth variety of a divisor with normal crossings.

**PROPOSITION 1.2** *Let  $j$  be the open immersion of  $T_N$  into a toric variety  $X$ , and denote by  $j^{\text{an}} : (T_N)^{\text{an}} \rightarrow X^{\text{an}}$  the corresponding open immersion of complex analytic spaces. Then we have a canonical quasi-isomorphism*

$$\mathbf{R}j_*^{\text{an}} \mathbf{C}_{(T_N)^{\text{an}}} \simeq \left( \Omega_X(\log D) \right)^{\text{an}} = (\mathcal{O}_X)^{\text{an}} \otimes_{\mathbf{Z}} \bigwedge^\bullet M$$

and canonical isomorphisms

$$H^\bullet((T_N)^{\text{an}}, \mathbf{C}) = \mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D)) = \mathbf{C} \otimes_{\mathbf{Z}} \bigwedge^\bullet M,$$

where  $\mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D))$  is the hypercohomology group of the complex  $\Omega_X^\bullet(\log D)$  consisting of  $\mathcal{O}_X$ -modules.

PROOF. We may assume  $X$  to be smooth, since  $\Omega_X^\bullet(\log D) = \mathcal{O}_X \otimes_{\mathbf{Z}} \bigwedge^\bullet M$  and since toric singularities are rational so that  $\mathbf{R}f_*\mathcal{O}_{X'} = \mathcal{O}_X$  holds for any equivariant resolution of singularities  $f : X' \rightarrow X$  (see, for instance, [16, Cor. 3.9] and Ishida [13, Cor. 3.3]). Consequently,  $D := X \setminus T_N$  is a divisor with simple normal crossings on a smooth  $X$  as in the situation dealt with by Grothendieck and Deligne. Let us repeat their proof here for convenience.

For the proof of the first assertion, we may obviously assume  $X$  to be affine as well, hence  $j^{\text{an}}$  is a Stein morphism. By the Poincaré lemma, we thus have

$$\mathbf{R}j_*^{\text{an}} \mathbf{C}_{(T_N)^{\text{an}}} = \mathbf{R}j_*^{\text{an}} (\Omega_{T_N}^\bullet)^{\text{an}} = j_*^{\text{an}} (\Omega_{T_N}^\bullet)^{\text{an}}.$$

The term on the extreme right hand side is canonically quasi-isomorphic to  $(\Omega_X^\bullet(\log D))^{\text{an}}$  by Deligne [3, Lemma 6.9].

As for the second assertion, note that the equality  $H^\bullet((T_N)^{\text{an}}, \mathbf{C}) = \bigwedge^\bullet M_{\mathbf{C}}$  itself is well-known, where  $M_{\mathbf{C}} := M \otimes_{\mathbf{Z}} \mathbf{C}$ . To show the whole set of the equalities, we start with the consequence

$$H^\bullet((T_N)^{\text{an}}, \mathbf{C}) = \mathbf{H}^\bullet(X^{\text{an}}, \mathbf{R}j_*^{\text{an}} \mathbf{C}_{(T_N)^{\text{an}}}) = \mathbf{H}^\bullet(X^{\text{an}}, (\Omega_X^\bullet(\log D))^{\text{an}})$$

of the first assertion. The term on the extreme right hand side is canonically isomorphic to the algebraic hypercohomology group  $\mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D))$  by Deligne [3, Thm. 6.2]. Since  $\Omega_X^\bullet(\log D) = \mathcal{O}_X \otimes_{\mathbf{Z}} \bigwedge^\bullet M$  has  $\mathcal{O}_X$ -coherent components, the algebraic hypercohomology group in question coincides with the cohomology group of the single complex associated to the Čech double complex  $\check{C}^\bullet(\mathcal{U}, \Omega_X^\bullet(\log D))$  with respect to the  $T_N$ -stable affine open covering  $\mathcal{U} := \{U_\sigma \mid \sigma \in \Delta\}$  with  $U_\sigma := \text{Spec}(\mathbf{C}[M \cap \sigma^\vee])$ . Moreover,  $T_N$  has a canonical algebraic action on the Čech double complex, which gives rise to an eigenspace decomposition

$$\check{C}^\bullet(\mathcal{U}, \Omega_X^\bullet(\log D)) = \bigoplus_{m \in M} \check{C}^\bullet(\mathcal{U}, \Omega_X^\bullet(\log D))_m$$

with respect to the characters  $m \in M$  of  $T_N$ . As a result, we have an eigenspace decomposition

$$\mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D)) = \bigoplus_{m \in M} \mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D))_m$$

for the hypercohomology group as well.

For  $m \neq 0$  we have  $\mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D))_m = 0$ , since the  $m$ -th component  $d_m$  of the exterior differentiation is the exterior multiplication by  $m$ , hence is exact.

On the other hand, for  $m = 0$  we have  $d_0 = 0$ , hence the Čech double complex is a direct sum of the complexes  $\check{C}^\bullet(\mathcal{U}, \Omega_X^p(\log D))_0$  for  $0 \leq p \leq r$ . Since  $\Omega_X^p(\log D) = \mathcal{O}_X \otimes_{\mathbf{Z}} \wedge^p M$  and since

$$H^l(X, \mathcal{O}_X)_0 = \begin{cases} \mathbf{C} & l = 0 \\ 0 & l \neq 0. \end{cases}$$

as we recall later in Lemma 3.2, we conclude that

$$\mathbf{H}^\bullet(X, \Omega_X^\bullet(\log D))_0 = H^0(X, \Omega_X^\bullet(\log D))_0 = \bigwedge^\bullet M_{\mathbf{C}}.$$

q.e.d.

Applying Proposition 1.2 to the immersion  $j_\sigma : \text{orb}(\sigma) \rightarrow X$  for each  $\sigma \in \Delta$ , we get the following:

**COROLLARY 1.3** *For each fixed  $q$ , we have a canonical quasi-isomorphism*

$$\left(\mathcal{L}_X^{\bullet, q}\right)^{\text{an}} \simeq \bigoplus_{\sigma \in \Delta(q)} \mathbf{R}(j_\sigma)_*^{\text{an}} \mathbf{C}_{\text{orb}(\sigma)}[-q],$$

where  $[-q]$  denotes the degree shift to the right by  $q$ .

## 2 Ishida's complexes

Let  $\Delta$  be a finite fan for a free  $\mathbf{Z}$ -module  $N$  of rank  $r$ , and for  $0 \leq q \leq r$  denote  $\Delta(q) := \{\sigma \in \Delta \mid \dim \sigma = q\}$  as before.

For each integer  $p$  with  $0 \leq p \leq r$ , *Ishida's  $p$ -th complex*  $C^\bullet(\Delta, \Lambda^p)$  of  $\mathbf{Z}$ -modules is defined as follows:

$$C^q(\Delta, \Lambda^p) := \bigoplus_{\sigma \in \Delta(q)} \bigwedge^{p-q}(M \cap \sigma^\perp) \quad \text{if } 0 \leq q \leq p,$$

and  $C^q(\Delta, \Lambda^p) = 0$  otherwise. For  $\sigma \in \Delta(q)$  and  $\tau \in \Delta(q+1)$ , the  $(\sigma, \tau)$ -component of the coboundary map

$$\delta : C^q(\Delta, \Lambda^p) = \bigoplus_{\sigma \in \Delta(q)} \bigwedge^{p-q}(M \cap \sigma^\perp) \longrightarrow C^{q+1}(\Delta, \Lambda^p) = \bigoplus_{\tau \in \Delta(q+1)} \bigwedge^{p-q-1}(M \cap \tau^\perp)$$

is defined to be 0 if  $\sigma$  is not a face of  $\tau$ . On the other hand, if  $\sigma$  is a face of  $\tau$ , then a primitive element  $n \in N$  is uniquely determined modulo  $N \cap \mathbf{R}\sigma$  so that  $\tau + (-\sigma) =$

$\mathbf{R}_{\geq 0}n + \mathbf{R}\sigma$ . The  $(\sigma, \tau)$ -component of  $\delta$  in this case is defined to be the interior product with respect to this  $n$ . Namely, the element  $m_1 \wedge m_2 \wedge \cdots \wedge m_{p-q}$  with  $m_1 \in M \cap \sigma^\perp$  and  $m_2, \dots, m_{p-q} \in M \cap \tau^\perp$  is sent to  $\langle m_1, n \rangle m_2 \wedge \cdots \wedge m_{p-q}$ . As Ishida [12, Prop. 1.6] showed,  $\delta \circ \delta = 0$  holds so that  $C^\bullet(\Delta, \Lambda^p)$  is a complex of  $\mathbf{Z}$ -modules. We denote its cohomology group by  $H^\bullet(\Delta, \Lambda^p)$ . We will be mainly concerned with their scalar extensions  $C^\bullet(\Delta, \Lambda^p)_{\mathbf{Q}}$ ,  $C^\bullet(\Delta, \Lambda^p)_{\mathbf{C}}$ ,  $H^\bullet(\Delta, \Lambda^p)_{\mathbf{Q}}$ ,  $H^\bullet(\Delta, \Lambda^p)_{\mathbf{C}}$  to  $\mathbf{Q}$  and  $\mathbf{C}$ .

By definition, we have  $H^q(\Delta, \Lambda^p) = 0$  unless  $0 \leq q \leq p$ .

REMARK. In [16, §3.2],  $C^\bullet(\Delta, \Lambda^p)$  and  $H^\bullet(\Delta, \Lambda^p)$  were denoted by  $C^\bullet(\Delta; p)$  and  $H^\bullet(\Delta; p)$ , respectively.

As in [17], we can define a similar complex  $C^\bullet(\Pi, \mathcal{G}_p)$  of  $\mathbf{R}$ -vector spaces for a *simplicial* polyhedral cone decomposition  $\Pi$  of an  $\mathbf{R}$ -vector space endowed with a marking for each one-dimensional cone in  $\Pi$ . Note, however, that unless a lattice  $N$  is given as in the case of a fan, we cannot define the coboundary map in the case of a *non-simplicial* convex polyhedral cone decomposition  $\Pi$  even if it is endowed with a marking. It is crucial that for a codimension one face  $\sigma$  of  $\tau$  in a fan, a primitive element  $n \in N$  is uniquely determined modulo  $N \cap \mathbf{R}\sigma$  so that  $\tau + (-\sigma) = \mathbf{R}_{\geq 0}n + \mathbf{R}\sigma$  holds as above, regardless of whether  $\tau$  is simplicial or not.

The following result is slightly stronger than [16, Lemma 3.7], and the proof is similar to that for [17, Prop. 3.5], which concerns analogous  $\mathbf{R}$ -coefficient cohomology groups for the simplicial polyhedral cone decomposition consisting of the faces of a simplicial cone in a finite dimensional  $\mathbf{R}$ -vector space:

PROPOSITION 2.1 *Let  $\pi$  be a simplicial rational polyhedral cone in  $N_{\mathbf{R}}$ . Then for each  $0 \leq p \leq r$ , the cohomology group of Ishida's  $p$ -th complex for the fan  $\Gamma_\pi$  consisting of all the faces of  $\pi$  satisfies*

$$H^q(\Gamma_\pi, \Lambda^p)_{\mathbf{Q}} := H^q(\Gamma_\pi, \Lambda^p) \otimes_{\mathbf{Z}} \mathbf{Q} = \begin{cases} \Lambda^p(M_{\mathbf{Q}} \cap \pi^\perp) & q = 0 \\ 0 & q \neq 0, \end{cases}$$

where  $M_{\mathbf{Q}} := M \otimes_{\mathbf{Z}} \mathbf{Q}$ .

### 3 The algebraic de Rham theorem

In this section, we denote by  $X := T_N \text{emb}(\Delta)$  the  $r$ -dimensional toric variety over  $\mathbf{C}$  corresponding to a finite fan  $\Delta$  for  $N \cong \mathbf{Z}^r$ . For simplicity, we again denote the corresponding complex analytic space by  $\mathcal{X} := X^{\text{an}}$ .



**PROPOSITION 3.1** *For an arbitrary fan  $\Delta$  which need not be complete nor simplicial, the hypercohomology group of the logarithmic complex  $\mathcal{L}_X^\bullet$  has a direct sum decomposition*

$$\mathbf{H}^l(X, \mathcal{L}_X^\bullet) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for each } l.$$

**PROOF.** Consider the  $T_N$ -stable affine open covering  $\mathcal{U} := \{U_\sigma \mid \sigma \in \Delta\}$  of  $X$  with  $U_\sigma := \text{Spec}(\mathbf{C}[M \cap \sigma^\vee])$ . We know that (cf. [16])

$$U_{\sigma_0} \cap U_{\sigma_1} \cap \cdots \cap U_{\sigma_q} = U_{\sigma_0 \cap \sigma_1 \cap \cdots \cap \sigma_q} \quad \text{for all } \sigma_0, \sigma_1, \dots, \sigma_q \in \Delta.$$

Since each component of  $\mathcal{L}_X^\bullet$  is a coherent  $\mathcal{O}_X$ -module, the hypercohomology group  $\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)$  coincides with the cohomology group of the single complex associated to the Čech double complex  $\check{C}^\bullet(\mathcal{U}, \mathcal{L}_X^\bullet)$  with respect to the affine open covering  $\mathcal{U}$ .

Because of the canonical algebraic action of the algebraic torus  $T_N$  on  $\check{C}^\bullet(\mathcal{U}, \mathcal{L}_X^\bullet)$ , we have the eigenspace decomposition

$$\check{C}^\bullet(\mathcal{U}, \mathcal{L}_X^\bullet) = \bigoplus_{m \in M} \check{C}^\bullet(\mathcal{U}, \mathcal{L}_X^\bullet)_m \quad \text{hence} \quad \mathbf{H}^\bullet(\mathcal{U}, \mathcal{L}_X^\bullet) = \bigoplus_{m \in M} \mathbf{H}^\bullet(\mathcal{U}, \mathcal{L}_X^\bullet)_m$$

with respect to the characters  $m \in M$ . Let us consider the triple complex

$$\check{C}_m^{\bullet, \bullet, \bullet} := \check{C}^\bullet(\mathcal{U}, \mathcal{L}_X^{\bullet, \bullet})_m \quad \text{with} \quad \check{C}_m^{l, p, q} := \check{C}^l(\mathcal{U}, \mathcal{L}_X^{p, q})_m.$$

The differential for  $l$  is the  $m$ -th component  $\check{\delta}_m$  of the Čech coboundary  $\check{\delta}$ , while those for  $p$  and  $q$  are the  $m$ -th components  $(d_I)_m$  and  $(d_{II})_m$  of  $d_I$  and  $d_{II}$ , respectively.

$\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)_m$  is the cohomology group of the single complex associated to  $\check{C}_m^{\bullet, \bullet, \bullet}$ . For  $m \neq 0$ , we have  $\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)_m = 0$ , hence  $\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet) = \mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)_0$  is  $T_N$ -invariant. Indeed, since  $(d_I)_m$  is the exterior multiplication by  $m$ , the complex  $(\check{C}_m^{l, \bullet, q}, (d_I)_m)$  is acyclic for all  $l$  and  $q$ . Consequently, the cohomology group vanishes for the single complex associated to the triple complex  $\check{C}_m^{\bullet, \bullet, \bullet}$ .

On the other hand, for  $m = 0$  we have  $(d_I)_0 = 0$ , hence  $\check{C}_0^{\bullet, \bullet, \bullet} = \bigoplus_p \check{C}_0^{\bullet, p, \bullet}$  is a direct sum of double complexes. Since  $\mathcal{L}_X^{p, q}$  is a direct sum of coherent sheaves of the form  $\mathcal{O}_{V(\sigma)}$ , we see by Lemma 3.2 below that the cohomology group of the single complex associated to  $\check{C}_0^{\bullet, p, \bullet}$  coincides with that of  $H^0(X, \mathcal{L}_X^{p, \bullet})_0 = C^\bullet(\Delta, \Lambda^p)_{\mathbf{C}}$ . Consequently, the cohomology group  $\mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet) = \mathbf{H}^\bullet(X, \mathcal{L}_X^\bullet)_0$  of the single complex associated to  $\check{C}_0^{\bullet, \bullet, \bullet}$  is the direct sum of  $H^\bullet(\Delta, \Lambda^p)_{\mathbf{C}}$  as claimed. q.e.d.

**LEMMA 3.2** *For any toric variety  $V$  with respect to an algebraic torus  $T_N$ , the  $T_N$ -invariant part of the cohomology group for the structure sheaf  $\mathcal{O}_V$  satisfies*

$$H^l(V, \mathcal{O}_V)_0 = \begin{cases} \mathbf{C} & l = 0 \\ 0 & l \neq 0. \end{cases}$$

PROOF. Demazure and Danilov gave a general description of the eigenspace  $H^*(V, L)_m$ , with respect to a character  $m$ , of the  $T_N$ -action on the cohomology group of an equivariant line bundle  $L$  on  $V$  in terms of the corresponding support function (see, for instance, [16, Thm. 2.6]). The present lemma is the special case  $L = \mathcal{O}_V$  and  $m = 0$ .  
q.e.d.

We are now ready to state a generalization, in the toric context, of the algebraic de Rham theorem due to Grothendieck [11]. An algebro-geometric proof valid in the case of complete nonsingular fans can be found in Danilov [2] and in [16, Theorem 3.11].

**THEOREM 3.3** (The algebraic de Rham theorem) *For a simplicial fan  $\Delta$  which need not be complete, we have the following for each  $l$ :*

$$H^l(\mathcal{X}, \mathbf{C}) = \mathbf{H}^l(\mathcal{X}, (\mathcal{L}_X^\bullet)^{\text{an}}) = \mathbf{H}^l(X, \mathcal{L}_X^\bullet) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}},$$

where the second term from the left is the corresponding analytic hypercohomology group.

PROOF. The first equality follows from Theorem 1.1, while the third is a consequence of Proposition 3.1. To show the second equality, we mimic the proof due to Grothendieck [11] in the case of the usual de Rham complex on smooth algebraic varieties.

There is a canonical homomorphism from the spectral sequence

$$E_1^{p,q} := H^q(X, \mathcal{L}_X^p) \implies \mathbf{H}^{p+q}(X, \mathcal{L}_X^\bullet)$$

for the algebraic hypercohomology to the spectral sequence

$$E_1^{p,q} := H^q(\mathcal{X}, (\mathcal{L}_X^p)^{\text{an}}) \implies \mathbf{H}^{p+q}(\mathcal{X}, (\mathcal{L}_X^\bullet)^{\text{an}})$$

for the complex analytic hypercohomology. To show that the homomorphism between the hypercohomology groups is an isomorphism, we may assume  $X$  to be affine, since the algebraic (resp. complex analytic) hypercohomology group is the cohomology group of the single complex associated to the Čech double complex with respect to the affine open covering  $\mathcal{U}$  (resp. the corresponding Stein open covering  $(\mathcal{U})^{\text{an}}$ ) as in the proof of Proposition 3.1.

We thus assume  $X = U_\pi := \text{Spec}(\mathbf{C}[M \cap \pi^\vee])$  for a simplicial rational cone  $\pi \subset N_{\mathbf{R}}$ , hence  $\Delta = \Gamma_\pi$ .

By Propositions 3.1 and 2.1, we have

$$\mathbf{H}^l(U_\pi, \mathcal{L}_{U_\pi}^\bullet) = \bigoplus_{p+q=l} H^q(\Gamma_\pi, \Lambda^p)_{\mathbf{C}} = \bigwedge^l (M \cap \pi^\perp)_{\mathbf{C}}$$

for each  $l$ . We are done in view of

$$\mathbf{H}^l((U_\pi)^{\text{an}}, (\mathcal{L}_{U_\pi})^{\text{an}}) = H^l((U_\pi)^{\text{an}}, \mathbf{C}) = \bigwedge^l (M \cap \pi^\perp)_{\mathbf{C}}$$

by Theorem 1.1 and Danilov [2, Lemma 12.3].

q.e.d.

**REMARK.** Suppose  $\Delta$  is an *arbitrary* finite fan which need not be simplicial nor complete. In view of the Verdier duality [24] and Ishida's result mentioned in the remark after Theorem 1.1, we have

$$\mathbf{H}^l(\mathcal{X}, (\mathcal{L}_X)^{\text{an}}) = \mathbf{H}^l(\mathcal{X}, \mathcal{D}_X[-2r]) = \text{Hom}_{\mathbf{C}}(H_c^{2r-l}(\mathcal{X}, \mathbf{C}), \mathbf{C}),$$

where the extreme right hand side is the dual of the cohomology group with compact support. Consequently, if we can show

$$\text{Hom}_{\mathbf{C}}(H_c^{2r-l}((U_\pi)^{\text{an}}, \mathbf{C}), \mathbf{C}) = \bigoplus_{p+q=l} H^q(\Gamma_\pi, \Lambda^p)_{\mathbf{C}} \quad \text{with} \quad U_\pi := T_N \text{emb}(\Gamma_\pi)$$

for an arbitrary strongly convex rational polyhedral cone  $\pi \subset N_{\mathbf{R}}$  which need not be simplicial, then Theorem 3.3 could be generalized for an *arbitrary* fan in the form

$$\text{Hom}_{\mathbf{C}}(H_c^{2r-l}(\mathcal{X}, \mathbf{C}), \mathbf{C}) = \mathbf{H}^l(\mathcal{X}, (\mathcal{L}_X)^{\text{an}}) = \mathbf{H}^l(X, \mathcal{L}_X) = \bigoplus_{p+q=l} H^q(\Delta, \Lambda^p)_{\mathbf{C}}.$$

## 4 Vanishing theorems

Analogues of the following were proved in Danilov [2] and [16, Theorem 3.11] for complete nonsingular fans by means of the algebraic de Rham theorem, and then directly in [17, Theorem 4.1] for complete simplicial polyhedral cone decompositions endowed with markings:

**PROPOSITION 4.1** *Let  $\Delta$  be a simplicial and complete fan for  $N \cong \mathbf{Z}^r$ . Then for all  $p$  with  $0 \leq p \leq r$  we have*

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \quad \text{for } q \neq p.$$

*Moreover for all  $p$ , we have a perfect pairing*

$$H^p(\Delta, \Lambda^p)_{\mathbf{Q}} \times H^{r-p}(\Delta, \Lambda^{r-p})_{\mathbf{Q}} \longrightarrow H^r(\Delta, \Lambda^r)_{\mathbf{Q}} \cong \bigwedge^r M_{\mathbf{Q}}.$$

*Consequently, the corresponding toric variety  $X := T_N \text{emb}(\Delta)$  satisfies  $H^l(X^{\text{an}}, \mathbf{C}) = 0$  for odd  $l$ , while*

$$H^{2p}(X^{\text{an}}, \mathbf{C}) = H^p(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for } 0 \leq p \leq r.$$

PROOF. We just indicate necessary modifications of the proof of [17, Theorem 4.1].

We introduce a first quadrant double complex  $(K^{\cdot,\cdot}, d', d'')$  as follows: For nonnegative integers  $i$  and  $j$ , let

$$K^{i,j} := \bigoplus_{\varphi \in \Delta(r-i)} \bigoplus_{\substack{\sigma \in \Delta(j) \\ \sigma \prec \varphi}} \bigwedge^{p-j} (M \cap \sigma^\perp) \otimes_{\mathbf{Z}} (\det \varphi)^{-1},$$

where  $\det \varphi := \bigwedge^{\dim \varphi} (N \cap \mathbf{R}\varphi)$  is the orientation  $\mathbf{Z}$ -module of rank one, and  $(\det \varphi)^{-1}$  is its dual  $\mathbf{Z}$ -module. If  $\psi \in \Delta(k-1)$  is a facet of  $\varphi \in \Delta(k)$ , we have mutually dual nonzero orientation  $\mathbf{Z}$ -homomorphisms

$$\det \psi \longrightarrow \det \varphi \quad \text{and} \quad (\det \varphi)^{-1} \longrightarrow (\det \psi)^{-1}$$

in the following manner: A primitive element  $n \in N$  is uniquely determined modulo  $N \cap \mathbf{R}\psi$  so that  $\varphi + (-\psi) = \mathbf{R}_{\geq 0}n + \mathbf{R}\psi$ . Then  $n_1 \wedge \cdots \wedge n_{k-1} \in \det \psi$  is sent to  $n \wedge n_1 \wedge \cdots \wedge n_{k-1} \in \det \varphi$ .

For  $\Delta(j) \ni \sigma \prec \varphi \in \Delta(r-i)$  and  $\Delta(j) \ni \sigma' \prec \psi \in \Delta(r-i-1)$ , we define the component of  $d' : K^{i,j} \rightarrow K^{i+1,j}$  with respect to  $(\varphi, \sigma)$  and  $(\psi, \sigma')$  to be nonzero only when  $\varphi \succ \psi \succ \sigma = \sigma'$  and to be equal to  $(-1)^j$  times the dual orientation  $\mathbf{Z}$ -homomorphism  $(\det \varphi)^{-1} \rightarrow (\det \psi)^{-1}$  tensored with the identity map for  $\bigwedge^{p-j}(M \cap \sigma^\perp)$ . On the other hand, for  $\Delta(j) \ni \sigma \prec \varphi \in \Delta(r-i)$  and  $\Delta(j+1) \ni \tau \prec \varphi' \in \Delta(r-i)$ , we define the component of  $d'' : K^{i,j} \rightarrow K^{i,j+1}$  with respect to  $(\varphi, \sigma)$  and  $(\varphi', \tau)$  to be nonzero only when  $\varphi = \varphi' \succ \tau \succ \sigma$  and to be equal to the tensor product of the homomorphism  $\bigwedge^{p-j}(M \cap \sigma^\perp) \rightarrow \bigwedge^{p-j-1}(M \cap \tau^\perp)$  appearing in the definition of Ishida's  $p$ -th complex, with the identity map for  $(\det \varphi)^{-1}$ . It is easy to show that  $(d')^2 = (d'')^2 = d'd'' + d''d' = 0$ , hence  $K^{\cdot,\cdot}$  is a double complex of  $\mathbf{Z}$ -modules.

Its scalar extension  $K_{\mathbf{Q}}^{\cdot,\cdot}$  to  $\mathbf{Q}$  turns out to satisfy

$$H_1^i(K_{\mathbf{Q}}^{\cdot,j}) = \begin{cases} C^j(\Delta, \Lambda^p)_{\mathbf{Q}} & i = 0 \\ 0 & i \neq 0, \end{cases}$$

since

$$H^{r-i}(\text{Star}_\sigma(\Delta), \mathbf{Q}) = \begin{cases} \mathbf{Q} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

for all  $\sigma \in \Delta(j)$ . Hence we get

$$H_{\text{II}}^j(H_1^i(K_{\mathbf{Q}}^{\cdot,\cdot})) = \begin{cases} H^j(\Delta, \Lambda^p)_{\mathbf{Q}} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

Consequently, by one of the two spectral sequences for the double complex, we see that the associated single complex  $(K_{\mathbf{Q}}^{\cdot,\cdot}, d' + d'')$  has the cohomology group

$$H^k(K_{\mathbf{Q}}^{\cdot,\cdot}) = H^k(\Delta, \Lambda^p)_{\mathbf{Q}} \quad \text{for all } k.$$

On the other hand, for each fixed  $i$ , we have an isomorphism of complexes

$$K_{\mathbf{Q}}^{i,\cdot} = \bigoplus_{\varphi \in \Delta(r-i)} \left( C^*(\Gamma_{\varphi}, \Lambda^p) \otimes_{\mathbf{Z}} (\det \varphi)^{-1} \right)_{\mathbf{Q}},$$

the coboundary map for the left hand side being  $d''$ , while that for the right hand side is  $\delta \otimes \text{id}$ . Thus by Proposition 2.1, we get

$$H_{\Pi}^j(K_{\mathbf{Q}}^{i,\cdot}) = \begin{cases} \bigoplus_{\varphi \in \Delta(r-i)} \left( (\Lambda^p(M \cap \varphi^{\perp})) \otimes_{\mathbf{Z}} (\det \varphi)^{-1} \right)_{\mathbf{Q}} & j = 0 \\ 0 & j \neq 0, \end{cases}$$

hence

$$H_{\Pi}^i(H_{\Pi}^j(K_{\mathbf{Q}}^{i,\cdot})) = 0 \quad \text{for } j \neq 0.$$

Consequently, we have

$$H^k(\Delta, \Lambda^p)_{\mathbf{Q}} = H^k(K_{\mathbf{Q}}^{\cdot}) = H_{\Pi}^k(H_{\Pi}^0(K_{\mathbf{Q}}^{\cdot}))$$

for all  $k$ . The extreme left hand side is nonzero only when  $0 \leq k \leq p$ , while the extreme right hand side is nonzero only when  $p \leq k$ .

The asserted perfect pairing is a consequence of the canonical identification, as in [17, Thm. 4.1], of the  $\mathbf{Q}$ -dual of  $H^p(\Delta, \Lambda^p)_{\mathbf{Q}} = H_{\Pi}^p(H_{\Pi}^0(K_{\mathbf{Q}}^{\cdot}))$  with  $(\Lambda^r N_{\mathbf{Q}}) \otimes_{\mathbf{Q}} H^{r-p}(\Delta, \Lambda^{r-p})_{\mathbf{Q}}$ . The rest of the assertion follows from Theorem 3.3. q.e.d.

The following is an important generalization, due to Ishida, of our earlier result stated as Corollary 4.4 later.

**THEOREM 4.2** (Ishida) *Let  $\Delta$  be a finite simplicial fan for  $N \cong \mathbf{Z}^r$  which may not be complete. If there exist a finite complete simplicial fan  $\tilde{\Delta}$  and a  $\rho \in \tilde{\Delta}(1)$  such that  $\Delta = \tilde{\Delta} \setminus \text{Star}_{\rho}(\tilde{\Delta})$ , then*

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \quad \text{for all } q \neq p.$$

*Consequently, the corresponding toric variety  $X := T_N \text{emb}(\Delta)$  satisfies  $H^l(X^{\text{an}}, \mathbf{C}) = 0$  for odd  $l$ , while*

$$H^{2p}(X^{\text{an}}, \mathbf{C}) = H^p(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for } 0 \leq p \leq r.$$

**PROOF.** Since  $\text{Star}_{\rho}(\tilde{\Delta})$  (resp.  $\Delta$ ) is a star closed subset (resp. a subcomplex) of  $\tilde{\Delta}$ , we have an exact sequence of complexes

$$0 \longrightarrow C^*(\text{Star}_{\rho}(\tilde{\Delta}), \Lambda^p) \longrightarrow C^*(\tilde{\Delta}, \Lambda^p) \longrightarrow C^*(\Delta, \Lambda^p) \longrightarrow 0.$$

Consider the projection  $N \rightarrow \bar{N} := N/\mathbf{Z}(N \cap \rho)$ . For each  $\sigma \in \text{Star}_\rho(\tilde{\Delta})$ , its image  $\bar{\sigma} := (\sigma + (-\rho))/\mathbf{R}\rho$  under the projection  $N_{\mathbf{R}} \rightarrow \bar{N}_{\mathbf{R}}$  is a strongly convex rational polyhedral cone, and

$$\bar{\Sigma} := \{\bar{\sigma} \mid \sigma \in \text{Star}_\rho(\tilde{\Delta})\}$$

is a finite complete simplicial fan for  $\bar{N}$ . The dual of  $\bar{N}$  is  $\bar{M} = M \cap \rho^\perp$ , hence  $M \cap \sigma^\perp = \bar{M} \cap \bar{\sigma}^\perp$  holds for each  $\sigma \in \text{Star}_\rho(\tilde{\Delta})$ . Consequently, we have an isomorphism of complexes

$$C^{\cdot-1}(\bar{\Sigma}, \Lambda^{p-1}) \xrightarrow{\sim} C^\cdot(\text{Star}_\rho(\tilde{\Delta}), \Lambda^p).$$

In view of Proposition 4.1 applied to  $\bar{\Sigma}$ , we thus have  $H^q(\text{Star}_\rho(\tilde{\Delta}), \Lambda^p)_{\mathbf{Q}} = 0$  for  $q \neq p$ . Again by Proposition 4.1 applied this time to  $\tilde{\Delta}$ , we have  $H^q(\tilde{\Delta}, \Lambda^p)_{\mathbf{Q}} = 0$  for  $q \neq p$ . Hence we get  $H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0$  for  $q \neq p-1, p$  as well as a long exact sequence

$$0 \longrightarrow H^{p-1}(\Delta, \Lambda^p)_{\mathbf{Q}} \longrightarrow H^p(\text{Star}_\rho(\tilde{\Delta}), \Lambda^p)_{\mathbf{Q}} \longrightarrow H^p(\tilde{\Delta}, \Lambda^p)_{\mathbf{Q}} \longrightarrow H^p(\Delta, \Lambda^p)_{\mathbf{Q}} \longrightarrow 0.$$

In particular, the vanishing of  $H^{p-1}(\Delta, \Lambda^p)_{\mathbf{Q}}$  is equivalent to the injectivity of

$$H^p(\text{Star}_\rho(\tilde{\Delta}), \Lambda^p)_{\mathbf{Q}} \rightarrow H^p(\tilde{\Delta}, \Lambda^p)_{\mathbf{Q}}.$$

To show the latter, we now introduce another finite complete simplicial fan  $\tilde{\Phi}$  for  $N$  containing  $\rho$  such that  $\text{Star}_\rho(\tilde{\Phi}) = \text{Star}_\rho(\tilde{\Delta})$ . This  $\tilde{\Phi}$  and  $\Phi := \tilde{\Phi} \setminus \text{Star}_\rho(\tilde{\Phi})$  turn out to be easier to handle, and we will be able to show  $H^q(\Phi, \Lambda^p)_{\mathbf{Q}} = 0$  for  $q \neq p$  in Lemma 4.3 below. Hence by the same argument as above applied to  $\tilde{\Phi}$  instead of  $\tilde{\Delta}$  we have the injectivity of

$$H^p(\text{Star}_\rho(\tilde{\Phi}), \Lambda^p)_{\mathbf{Q}} \longrightarrow H^p(\tilde{\Phi}, \Lambda^p)_{\mathbf{Q}}.$$

Clearly, there exists a finite simplicial complete fan  $\tilde{\Delta}'$  for  $N$  which is a subdivision of both  $\tilde{\Delta}$  and  $\tilde{\Phi}$  such that  $\text{Star}_\rho(\tilde{\Delta}') = \text{Star}_\rho(\tilde{\Delta}) = \text{Star}_\rho(\tilde{\Phi})$ .

We claim the injectivity of the canonical homomorphisms

$$H^p(\tilde{\Delta}, \Lambda^p)_{\mathbf{Q}} \longrightarrow H^p(\tilde{\Delta}', \Lambda^p)_{\mathbf{Q}} \quad \text{and} \quad H^p(\tilde{\Phi}, \Lambda^p)_{\mathbf{Q}} \longrightarrow H^p(\tilde{\Delta}', \Lambda^p)_{\mathbf{Q}}$$

induced by the subdivisions. Consequently, we would get the injectivity of

$$H^p(\text{Star}_\rho(\tilde{\Delta}), \Lambda^p)_{\mathbf{Q}} \rightarrow H^p(\tilde{\Delta}, \Lambda^p)_{\mathbf{Q}},$$

since the canonical homomorphism from

$$H^p(\text{Star}_\rho(\tilde{\Phi}), \Lambda^p)_{\mathbf{Q}} = H^p(\text{Star}_\rho(\tilde{\Delta}), \Lambda^p)_{\mathbf{Q}} = H^p(\text{Star}_\rho(\tilde{\Delta}'), \Lambda^p)_{\mathbf{Q}}$$

to  $H^p(\tilde{\Phi}, \Lambda^p)_{\mathbf{Q}}$  is injective as above.

As for the proof of the above claim, it obviously suffices to prove the injectivity of the scalar extension to  $\mathbf{C}$  of the canonical homomorphisms in question. Let  $\widetilde{X}$ ,  $\widetilde{X}'$ ,  $\widetilde{Y}$  be the complete toric varieties associated to the fans  $\widetilde{\Delta}$ ,  $\widetilde{\Delta}'$ ,  $\widetilde{\Phi}$ , respectively, and denote by  $f : \widetilde{X}' \rightarrow \widetilde{X}$  and  $g : \widetilde{X}' \rightarrow \widetilde{Y}$  the equivariant proper birational morphisms associated to the subdivisions of the fans. By Theorem 3.3 and Proposition 4.1, we have

$$\begin{aligned} H^p(\widetilde{\Delta}, \Lambda^p)_{\mathbf{C}} &= H^{2p}((\widetilde{X})^{\text{an}}, \mathbf{C}), \\ H^p(\widetilde{\Delta}', \Lambda^p)_{\mathbf{C}} &= H^{2p}((\widetilde{X}')^{\text{an}}, \mathbf{C}), \\ H^p(\widetilde{\Phi}, \Lambda^p)_{\mathbf{C}} &= H^{2p}((\widetilde{Y})^{\text{an}}, \mathbf{C}). \end{aligned}$$

The canonical homomorphisms in question coincide with

$$f^* : H^{2p}((\widetilde{X})^{\text{an}}, \mathbf{C}) \rightarrow H^{2p}((\widetilde{X}')^{\text{an}}, \mathbf{C}) \quad \text{and} \quad g^* : H^{2p}((\widetilde{Y})^{\text{an}}, \mathbf{C}) \rightarrow H^{2p}((\widetilde{X}')^{\text{an}}, \mathbf{C}),$$

which are well-known to be injective by  $f_* f^* = \text{id}$  and  $g_* g^* = \text{id}$ , where  $f_*$  and  $g_*$  are the direct images

$$f_* : H^{2p}((\widetilde{X}')^{\text{an}}, \mathbf{C}) \rightarrow H^{2p}((\widetilde{X})^{\text{an}}, \mathbf{C}) \quad \text{and} \quad g_* : H^{2p}((\widetilde{X}')^{\text{an}}, \mathbf{C}) \rightarrow H^{2p}((\widetilde{Y})^{\text{an}}, \mathbf{C}).$$

Here is how we define the new complete simplicial fan  $\widetilde{\Phi}$  for  $N$  which satisfies the required properties mentioned above:

$\Delta^{\flat} := \{\tau \in \Delta \mid \tau + \rho \in \widetilde{\Delta}\}$  is easily seen to be a fan for  $N$ . It can be thought of as the “boundary” of  $\Delta$  as well as the “link” of  $\rho$  in  $\widetilde{\Delta}$  so that  $\text{Star}_{\rho}(\widetilde{\Delta}) = \{\tau + \rho \mid \tau \in \Delta^{\flat}\}$ . The projection  $N \rightarrow \bar{N}$  induces a bijection from each  $\tau \in \Delta^{\flat}$  to its image  $\bar{\tau} \in \bar{\Sigma}$ . The cone  $-\rho$  in  $N_{\mathbf{R}}$  is not contained in  $\text{Star}_{\rho}(\widetilde{\Delta})$  nor  $\Delta^{\flat}$ . Hence

$$\widetilde{\Phi} := \{\tau + \rho \mid \tau \in \Delta^{\flat}\} \amalg \Delta^{\flat} \amalg \{\tau + (-\rho) \mid \tau \in \Delta^{\flat}\}$$

is a finite simplicial complete fan for  $N$  satisfying  $\text{Star}_{\rho}(\widetilde{\Phi}) = \text{Star}_{\rho}(\widetilde{\Delta})$ . Let  $\Phi^{\flat} := \Delta^{\flat}$  and

$$\Phi := \Phi^{\flat} \amalg \{\tau + (-\rho) \mid \tau \in \Phi^{\flat}\}.$$

We clearly have  $\Phi = \widetilde{\Phi} \setminus \text{Star}_{\rho}(\widetilde{\Phi})$  and  $\Phi^{\flat} = \{\tau \in \Phi \mid \tau + \rho \in \widetilde{\Phi}\}$ .

We are done in view of Lemma 4.3 below and Theorem 3.3 q.e.d.

**LEMMA 4.3** *In the notation above, we have  $H^*(\Phi, \Lambda^p)_{\mathbf{Q}} = H^*(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}}$ . In particular,*

$$H^q(\Phi, \Lambda^p)_{\mathbf{Q}} = 0 \quad \text{for all } q \neq p,$$

*hence the canonical homomorphism  $H^p(\text{Star}_{\rho}(\widetilde{\Phi}), \Lambda^p)_{\mathbf{Q}} \rightarrow H^p(\widetilde{\Phi}, \Lambda^p)_{\mathbf{Q}}$  is injective.*

PROOF. As we mentioned above,  $\Phi^b = \Delta^b$  is in one-to-one correspondence with  $\bar{\Sigma}$  by the map which sends  $\tau \in \Phi^b$  to its isomorphic image under the projection  $N_{\mathbf{R}} \rightarrow \bar{N}_{\mathbf{R}} = N_{\mathbf{R}}/\mathbf{R}\rho$ . On the other hand,  $\Phi^b(q)$  for each  $q$  is in one-to-one correspondence with  $\text{Star}_{-\rho}(\Phi)(q+1)$  by the map which sends  $\tau \in \Phi^b$  to  $\tau + (-\rho) \in \text{Star}_{-\rho}(\Phi)$ . We have  $\bar{M} \cap \bar{\tau}^\perp = M \cap (\tau + (-\rho))^\perp$  and an exact sequence

$$0 \longrightarrow (\bar{M} \cap \bar{\tau}^\perp)_{\mathbf{Q}} \longrightarrow (M \cap \tau^\perp)_{\mathbf{Q}} \longrightarrow \mathbf{Q} \longrightarrow 0,$$

where the second arrow from the right is the interior product with the unique primitive element  $-n(\rho)$  of  $N$  contained in the cone  $-\rho$ . As a result, we have an exact sequence

$$0 \longrightarrow \bigwedge^{p-q}(\bar{M} \cap \bar{\tau}^\perp)_{\mathbf{Q}} \longrightarrow \bigwedge^{p-q}(M \cap \tau^\perp)_{\mathbf{Q}} \longrightarrow \bigwedge^{p-q-1}(\bar{M} \cap \bar{\tau}^\perp)_{\mathbf{Q}} \longrightarrow 0$$

for each  $q$ , hence an exact sequence of complexes

$$0 \longrightarrow C^*(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}} \longrightarrow C^*(\Phi^b, \Lambda^p)_{\mathbf{Q}} \xrightarrow{\iota} C^*(\bar{\Sigma}, \Lambda^{p-1})_{\mathbf{Q}} \longrightarrow 0.$$

On the other hand, we have

$$C^q(\Phi, \Lambda^p) = C^q(\Phi^b, \Lambda^p) \oplus C^q(\text{Star}_{-\rho}(\Phi), \Lambda^p) = C^q(\Phi^b, \Lambda^p) \oplus C^{q-1}(\bar{\Sigma}, \Lambda^{p-1})$$

for each  $q$ . We see easily that  $C^*(\Phi, \Lambda^p)_{\mathbf{Q}}$  coincides with the mapping cone of the surjective homomorphism  $\iota : C^*(\Phi^b, \Lambda^p)_{\mathbf{Q}} \rightarrow C^*(\bar{\Sigma}, \Lambda^{p-1})_{\mathbf{Q}}$ , whose kernel is  $C^*(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}}$  as we saw above. Consequently, we have

$$H^*(\Phi, \Lambda^p)_{\mathbf{Q}} = H^*(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}}.$$

In particular, we have  $H^q(\Phi, \Lambda^p)_{\mathbf{Q}} = H^q(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}} = 0$  for  $q \neq p$  by Proposition 4.1. q.e.d.

**COROLLARY 4.4** *Let  $\Delta$  be a finite simplicial fan for  $N \cong \mathbf{Z}^r$  such that its support  $|\Delta|$  is convex of dimension  $r$ . Then*

$$H^q(\Delta, \Lambda^p)_{\mathbf{Q}} = 0 \quad \text{for all } q \neq p.$$

*Consequently, the corresponding toric variety  $X := T_N \text{emb}(\Delta)$  satisfies  $H^l(X^{\text{an}}, \mathbf{C}) = 0$  for odd  $l$ , while*

$$H^{2p}(X^{\text{an}}, \mathbf{C}) = H^p(\Delta, \Lambda^p)_{\mathbf{C}} \quad \text{for } 0 \leq p \leq r.$$

PROOF. By Proposition 4.1, we may assume that  $\Delta$  is not complete. Then there exists a primitive element  $n^\circ \in N$  such that  $-n^\circ$  is contained in the interior of  $|\Delta|$ . Let  $\rho := \mathbf{R}_{\geq 0}n^\circ$ . Denote by  $\Delta^b$  the subcomplex of  $\Delta$  consisting of those  $\sigma \in \Delta$  which are contained in the boundary of the convex cone  $|\Delta|$ . Obviously,

$$\tilde{\Delta} := \Delta \amalg \{\sigma + \rho \mid \sigma \in \Delta^b\}$$



is a finite complete simplicial fan for  $N$  such that  $\Delta = \tilde{\Delta} \setminus \text{Star}_\rho(\tilde{\Delta})$ . We are done by Theorem 4.2 and Theorem 3.3. q.e.d.

$\tilde{\Phi}$ ,  $\Phi$  and  $\Phi^\flat$  appearing in the proof of Theorem 4.2, Lemma 4.3 and Corollary 4.4 are of independent interest. Namely, let  $n_0$  be the primitive element in  $N$  such that  $\rho = \mathbf{R}_{\geq 0}(-n_0)$ , and choose a splitting  $N \cong \bar{N} \oplus \mathbf{Z}n_0$ . Then there exists a function  $\eta : \bar{N}_{\mathbf{R}} \rightarrow \mathbf{R}$ , which is  $\mathbf{Z}$ -valued on  $\bar{N}$  and piecewise linear with respect to the complete nonsingular fan  $\bar{\Sigma}$  for  $\bar{N}$  so that, in terms of the graph  $g : \bar{N}_{\mathbf{R}} \rightarrow N_{\mathbf{R}}$  of  $\eta$  defined by  $g(\bar{n}) := \bar{n} + \eta(\bar{n})n_0$  for each  $\bar{n} \in \bar{N}_{\mathbf{R}}$ , we have

$$\begin{aligned}\Phi^\flat &= \{g(\bar{\sigma}) \mid \bar{\sigma} \in \bar{\Sigma}\} \\ \Phi &= \Phi^\flat \coprod \{\tau + \mathbf{R}_{\geq 0}n_0 \mid \tau \in \Phi^\flat\} \\ \tilde{\Phi} &= \Phi \coprod \{\tau + \mathbf{R}_{\geq 0}(-n_0) \mid \tau \in \Phi^\flat\}.\end{aligned}$$

It is easy to see that  $T_N \text{emb}(\tilde{\Phi}) \rightarrow T_{\bar{N}} \text{emb}(\bar{\Sigma})$  is a  $\mathbf{P}_1(\mathbf{C})$ -bundle, while  $T_N \text{emb}(\Phi) \rightarrow T_{\bar{N}} \text{emb}(\bar{\Sigma})$  and  $T_N \text{emb}(\Phi^\flat) \rightarrow T_{\bar{N}} \text{emb}(\bar{\Sigma})$  are the associated  $\mathbf{C}$ -bundle and the associated  $\mathbf{C}^*$ -bundle, respectively.

As in [17, Thm. 4.3 and Prop. 4.4] in different notation and in Park [19], [20],  $\eta$  determines an element  $\bar{\eta} \in H^1(\bar{\Sigma}, \Lambda^1)$ , the multiplication by which induces a homomorphism

$$H^{p-1}(\bar{\Sigma}, \Lambda^{p-1})_{\mathbf{Q}} \longrightarrow H^p(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}}$$

for each  $p$ . Then we have the following:

**COROLLARY 4.5** *In the notation above, we have*

$$H^q(\Phi^\flat, \Lambda^p) = 0 \quad \text{for } q \neq p-1, p.$$

Moreover,  $H^{p-1}(\Phi^\flat, \Lambda^p)_{\mathbf{Q}}$  (resp.  $H^p(\Phi^\flat, \Lambda^p)_{\mathbf{Q}}$ ) coincides with the kernel (resp. cokernel) of the homomorphism

$$H^{p-1}(\bar{\Sigma}, \Lambda^{p-1})_{\mathbf{Q}} \longrightarrow H^p(\bar{\Sigma}, \Lambda^p)_{\mathbf{Q}}$$

induced by multiplication of the element  $\bar{\eta} \in H^1(\bar{\Sigma}, \Lambda^1)$ .

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